Mathematical Study of Weibull – New Weighted Exponential Distribution

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***Abstract*— A four Parameter Weibull - New Weighted Exponential distribution was proposed with the use of the Weibull Generalized family of distributions. The Weibull - New Weighted Exponential distribution has the ability to model failure rates which are quite common in medical sciences and reliability/survival Studies. Mathematical properties of the proposed distribution such as Moments, reliability analysis and order Statistics were derived. The Method of Maximum Likelihood Estimation was used in estimating the distribution Parameters.**

***Keywords*— Generalization, maximum likelihood estimation, Azallini, order statistics, Weibull Generalized family of distributions.**

1. Introduction

Numerous classical distributions have been extensively considered by researcher for analysing lifetime data set over the past decades. Different extensions of existing distributions have been considered for modelling data in several areas of real life data such as, actuarial, medical sciences, engineering and many more. However, in many of these applied areas, there is continuous need for extended forms of existing distributions. For that reason, several methods for generating new families of distributions have been studied. (Bourguignon, Silva and Cordeiro, 2014)

The Weibull, gamma and exponential distributions are common distributions that have been extensively used over years for modelling data in reliability and biological studies. Much attention has been given to these distributions in literature in modelling lifetime data. However, researchers continue to develop different extensions of the Weibull, gamma and exponential distribution that will suit different behaviour of real life data set.

The Weibull distribution is a very popular model and the distribution has been considered extensively over the past years for modelling data in reliability and engineering studies. It is generally adequate for modelling monotone hazard rates. Bourguignon et al., (2014)

Bourguignon et al., (2014) gave expression for generating Weibull family of distribution as follows:

Consider a positive continuous distribution G with density g and the Weibull CDF

$F(\mathrm{x}) = 1 - \mathrm{e}^{-\alpha x^{\beta}}$ (for x>0) with positives α and β.

Based on this density, by replacing x with $\frac{G(\mathrm{x})}{1-G(x)}$, the CDF family is defined by

$$F(x,\alpha,\beta) = \int_0^{\frac{G(\mathrm{x})}{1-G(x)}} \alpha\beta t^{\beta-1} e^{-\alpha t^{\beta}} dt \qquad 1$$

$$F(x,\alpha,\beta) = 1 - \exp\left\{-\alpha\left[\frac{G(\mathrm{x})}{1-G(x)}\right]^{\beta}\right\} \qquad x,\alpha,\beta > 0 \qquad 2$$

Where $G(\mathrm{x})$ CDF of baseline distribution and the family PDF is given as

$$f(x,\alpha,\beta) = \alpha\beta g(\mathrm{x}) \frac{[G(x)]^{\beta-1}}{[1-G(x)]^{\beta+1}} \exp\left\{-\alpha\left[\frac{G(\mathrm{x})}{1-G(x)}\right]^{\beta}\right\} \qquad 3$$

By using the exponential function in (3), we have

$$\exp\left\{-\alpha\left[\frac{G(\mathrm{x})}{1-G(x)}\right]^{\beta}\right\}=\sum_{k=0}^{\infty}\frac{(-1)^{k}\alpha^{k}}{k!}\left[\frac{G(\mathrm{x})}{1-G(\mathrm{x})}\right]^{k\beta}$$

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Substitute this expansion into (3), we have

$$f(\mathrm{x},\alpha,\beta)=\alpha\beta\,\mathrm{g}(\mathrm{x})\sum_{0}^{\infty}\frac{(-1)^{k}\alpha^{k}}{k!}\frac{[G(\mathrm{x})]^{\beta(\mathrm{k}+1)-1}}{[1-G(\mathrm{x})]^{\beta(\mathrm{k}+1)+1}}$$

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Aljarrah, Famoye and Lee, (2015) and Cordeiro, Edwin and Ramires, (2015) gave expressions for other forms of generalizing Weibull family of distribution.

Another expression of Equation (1) was given by Nasiru, and Luguterah, (2015) as follows

$$F(x)=\int_{0}^{\frac{1}{1-G(x)}} f(x)dx$$

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A 3-parameter probability distribution called Weibull Exponential distribution was proposed by Oguntunde, Balogun and Amina, (2015) using the concept of the Weibull Generalized family of distribution.

2. MATERIALS AND METHODS

Let consider distribution proposed by Oguntunde et al (2016) called new weighted exponential distribution as the baseline distribution on the weibull family of distribution as expressed in Equation (2) and (3)

The pdf and cdf of the new weighted exponential distribution by Oguntunde et al (2016) are defined as

$$\mathrm{g}(x)=(1+\lambda)\alpha\,\mathrm{e}^{-(1+\lambda)\alpha x}$$

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$$\mathrm{G}(x)=1-\mathrm{e}^{-(1+\lambda)\alpha x}$$

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To derive the cdf of the Weibull new weighted exponential distribution, the cdf of the new weighted exponential distribution as given in Equation (8) is inserted into Equation (2). The resulting expression is given as:

$$F(x)=1-e\left[-\theta\left[\frac{1-e^{-(1+\lambda)\alpha x}}{e^{-(1+\lambda)\alpha x}}\right]^{\beta}\right]$$

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The cdf can further be simplify as:

$$F(x)=1-e^{-\theta[e^{(1+\lambda)\alpha x}-1]^{\beta}}$$

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Also, the corresponding pdf is obtained by inserting the pdf and cdf of the new weighted exponential distribution into Equation (3). The resulting expression is given as:

$$f(x)=\alpha\beta(1+\lambda)\theta e^{-(1+\lambda)\alpha x}\left[\frac{(1-e^{-(1+\lambda)\alpha x})^{\beta-1}}{(e^{-(1+\lambda)\alpha x})^{\beta+1}}\right]e\left[-\theta\left[\frac{1-e^{-(1+\lambda)\alpha x}}{e^{-(1+\lambda)\alpha x}}\right]^{\beta}\right]$$

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The pdf can further be simplify as:

$$f(x)=\alpha\beta\theta(1+\lambda)(1-e^{-(1+\lambda)\alpha x})^{\beta-1}e^{(1+\lambda)\alpha\beta x-\theta[\mathrm{e}^{(1+\lambda)\alpha x}-1]^{\beta}}$$

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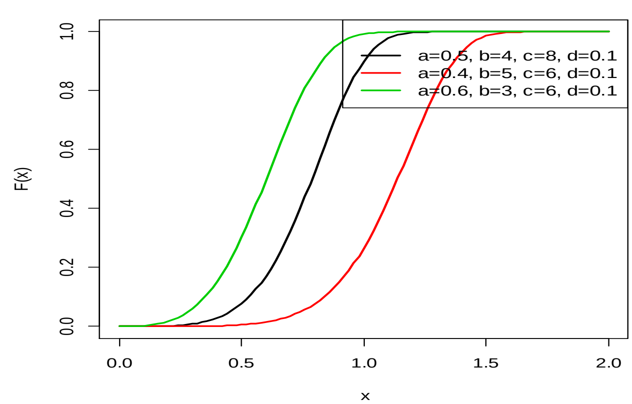

Figure 1: cdf of Weibull new weighted exponential distribution

$$where, \quad a=\alpha, b=\beta, c=\theta, d=\lambda$$

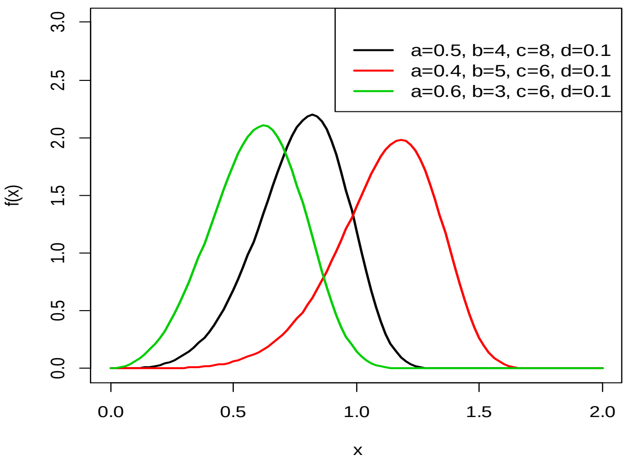

Figure 2: pdf of Weibull new weighted exponential distribution

$$where, \quad a=\alpha, b=\beta, c=\theta, d=\lambda$$

3. STATISTICAL PROPERTIES OF WEIBULL NEW WEIGHTED EXPONENTIAL DISTRIBUTION

I. RELIABILITY ANALYSIS:

The Survivor function indicates the probability that the event of interest has not yet occurred by time t; thus, if T denotes time until failure, $S(t)$ denotes probability of surviving beyond t.

The expression for Survival (or reliability) function is given as:

$$S(x)=1-F(x)$$

Therefore, the survival function for the WNWE distribution is:

$$S(x)=e^{-\theta[e^{(1+\lambda)\alpha x}-1]^{\beta}} \quad ; x,\alpha,\beta,\lambda,\theta>0 \qquad 13$$

II. HAZARD FUNCTION OF WNWE DISTRIBUTION

Hazard function is a conditional probability that the event of interest (device failure) will occur during the time t and dt under the condition that the device is safe until time t. hazard function is defined as the ratio of probability density function $f(x)$ to the survival function $S(x)$ given as:

$$h(x)=\frac{f(x)}{S(x)}$$

Therefore, the hazard function for the WNWE distribution is:

$$h(x)=\frac{\alpha\beta\theta(1+\lambda)(1-e^{-(1+\lambda)\alpha x})^{\beta-1}e^{(1+\lambda)\alpha\beta x-\theta[e^{(1+\lambda)\alpha x}-1]^{\beta}}}{e^{-\theta[e^{(1+\lambda)\alpha x}-1]^{\beta}}}$$

$$x,\alpha,\beta,\lambda,\theta>0 \qquad 14$$

III. REVERSED HAZARD FUNCTION OF WNWE DISTRIBUTION

The reverse hazard function is defined as the ratio of probability density function $f(x)$ to the distribution function $F(x)$ of a random variable, given by

$$r(x)=\frac{f(x)}{F(x)}$$

Therefore, the reversed hazard function for the WNWE distribution is:

$$r(x)=\frac{\alpha\beta\theta(1+\lambda)(1-e^{-(1+\lambda)\alpha x})^{\beta-1}e^{(1+\lambda)\alpha\beta x-\theta[e^{(1+\lambda)\alpha x}-1]^{\beta}}}{1-e^{-\theta[e^{(1+\lambda)\alpha x}-1]^{\beta}}}$$

$$; \quad x,\alpha,\beta,\lambda,\theta>0 \qquad 15$$

IV. MOMENTS

Some properties of the Weibull Generalized family of distributions can be directly obtained from those of the exponentiated Generalized family of distributions (Bourguignon *et al*., 2014). Therefore, the rth moment of X can be expressed as:

$$E(X^r)=\sum_{j,k=0}^{\infty}W_{j,k}E(Z_{j,k}^r) \qquad 16$$

Then, Weibull exponential distribution, $Z_{j,k}$ is the exponentiated exponential distribution with power parameter $\beta(\mathrm{k}+1)+\mathrm{j}-1$

V. ORDER STATISTICS

For a random sample $X_1,...,X_n$ the probability density function of the ith order statistic from a distribution function F(x) and probability density function f(x) is defined as

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}f(x)F(x)^{i-1}\left[1-F(x)\right]^{n-i}$$

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Inserting the pdf and cdf of the Weibull new weighted exponential distribution into the expression of probability density function of the ith order statistics. The resulting expression is given as

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}\left[\alpha\beta\theta(1+\lambda)(1-e^{-(1+\lambda)\alpha x})^{\beta-1}e^{(1+\lambda)\alpha\beta x-\theta[e^{(1+\lambda)\alpha x}-1]^{\beta}}\right] \times\left[1-e^{-\theta[e^{(1+\lambda)\alpha x}-1]^{\beta}}\right]^{i-1}\left[e^{-\theta[e^{(1+\lambda)\alpha x}-1]^{\beta}}\right]^{n-i}$$

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The pdf of the Minimum order statistic $X_{(1)}$ of Weibull new weighted exponential distribution is obtained as

$$f_1(x)=n\left[\alpha\beta\theta(1+\lambda)(1-e^{-(1+\lambda)\alpha x})^{\beta-1}e^{(1+\lambda)\alpha\beta x-\theta[e^{(1+\lambda)\alpha x}-1]^{\beta}}\right] \times\left[e^{-\theta[e^{(1+\lambda)\alpha x}-1]^{\beta}}\right]^{n-1}$$

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Also, the pdf of the Maximum order statistic $X_{(n)}$ of Weibull new weighted exponential distribution is obtained as

$$f_n(x)=n\left[\alpha\beta\theta(1+\lambda)(1-e^{-(1+\lambda)\alpha x})^{\beta-1}e^{(1+\lambda)\alpha\beta x-\theta[e^{(1+\lambda)\alpha x}-1]^{\beta}}\right] \times\left[1-e^{-\theta[e^{(1+\lambda)\alpha x}-1]^{\beta}}\right]^{n-1}$$

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VI. PARAMETER ESTIMATION

The Estimation of the Weibull new weighted exponential distribution is obtained using the Method of Maximum Likelihood Estimation (MLE) as follows:

Let $x_1, x_2,..., x_n$ be a random sample of size "n" from Weibull new weighted exponential distribution defined in Equation (12), the Likelihood function $L(x/\alpha,\lambda,\beta.\theta)$ is given by

$$L(x/\alpha,\lambda,\beta,\theta)=\prod_{i=1}^{n}f(\mathrm{x}_i/\alpha,\lambda)$$

let $l=\log L(x/\alpha,\lambda,\beta,\theta)$

$$l=n\log\alpha+n\log\beta+n\log\theta+n\log(1+\lambda)+(\beta-1)\sum_{i=1}^{n}\log[1-e^{-(1+\lambda)\alpha x_i}] +\sum_{i=1}^{n}\{(1+\lambda)\beta\alpha x_i-\theta[\mathrm{e}^{(1+\lambda)\alpha x_i}-1]^{\beta}\}$$

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The solution of the non-linear system of equations obtained by differentiating Equation (21) with respect to α, λ, β and θ gives the maximum likelihood estimates of the model parameters. Also, the solution can be obtained directly by using R software when data sets are available.

4. CONCLUSION

A four-parameter Weibull new weighted exponential distribution has been successfully derived and defined. The proposed distribution can be suitable for modelling lifetime data. The shape of model is unimodal. Some basic mathematical properties of proposed distribution such as survival function, hazard function, moment and order statistics are discussed. The method of maximum likelihood was considered for parameter estimation. The Weibull new weighted exponential distribution can be used as an alternative for a life testing model.

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