GEOMETRIC PROPERTIES OF A NEW SUBCLASS OF ANALYTIC FUNCTIONS BY MEANS OF CHEBYSHEV POLYNOMIALS

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***Abstract*— In this work, Opoola Differential Operator which is a generalization of Salagean Differential Operator and Al-Oboudi Differential Operator is used to define new subclasses of analytic univalent functions using subordination principle. Also, by means of Chebyshev polynomials of the second kind, coefficient estimates, upper bounds for the Fekete-Szego functional for the new subclasses defined are obtained.**

***Keywords*— Analytic, Univalent, Opoola differential operator, Chebyshev polynomials, coefficient estimates, Fekete-Szego functional.**

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I. INTRODUCTION

Let A be the class of functions analytic in the unit disk $\mathbb{U} = \{z \epsilon \mathbb{C} : |z| < 1\}$. $f(z)$ is said to be in the class S if

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

with the conditions $f(0) = 0,\ f'(0) = 1$.

Chebyshev polynomials which are of four kinds, are widely used in many areas like numerical solutions of ordinary and partial differential equations, analytic functions and so on. Many research journals are using the first and second kinds of Chebyshev polynomials because of it's orthogonality property. See [2-4], [9] and [12].

The Chebyshev polynomial of the second kind in t of degree n can be defined as:

$$U_n(t) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad t = \cos\theta, \quad t \in [-1,1]$$

The generating function for Chebyshev polynomial of the second kind is given as

$$H(z,t) = 1 + 2tz + (4t^2-1)z^2 + (8t^3-4t)z^3 + (16t^4-12t^2+1)z^4 + \cdots \qquad (1)$$

The Fekete-Szego inequality states that if

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \qquad (2)$$

is a univalent analytic function on the unit disk and $0 \le \lambda \le 1$, then

$$C(\lambda) = |a_3 - \lambda a_2^2| < 1 + 2e^{\left(\frac{-2\lambda}{1-\lambda}\right)}$$

For a class of functions in A and a real number λ, the Fekete-Szego problem is all about finding the best possible $C(\lambda)$ for functions in A. Some of the researchers that have worked in this area are: [10], [13] and [20].

Let f(z) and g(z) be analytic functions in the unit disk $\mathbb{U}$, then f(z) is subordinate to g(z) in $\mathbb{U}$ written as $f(z) \prec g(z)$, if there exist a function w(z) analytic in $\mathbb{U}$ with w(0)=0, |w(z)| < 1 which is called the Schwarz function such that f(z) = g(w(z)). If the function g is univalent in $\mathbb{U}$, then $f(z) \prec g(z)$; $z \in \mathbb{U} \Leftrightarrow$ f(0) = g(0) and $f(\mathbb{U}) \subset g(\mathbb{U})$. see[16].

A function $f(z) \in S$ of the form (2) is star-like with respect to the unit disk $\mathbb{U}$ if it maps a unt disk onto a star-like domain and it is denoted by S^*. A necessary and sufficient condition for $f(z)$ to be star-like is that

$$Re\left\{z\frac{f'(z)}{f(z)}\right\} > 0, \quad z \in \mathbb{U}$$

An analytic function $f(z)$ is convex if maps $\mathbb{U}$ conformally onto a convex domain and it is denoted by C. Equivalently, $f \in C$ if and only if it satisfies the following condition

$$Re\left\{1 + z\frac{f''(z)}{f'(z)}\right\} > 0, \quad z \in \mathbb{U}$$

II PRELIMINARIES

Lemma 1. [6]

If $p \in P$ with the series expansion $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots, z \in \mathbb{U}$ with $Re(p(z)) > 0$ and $p(0) = 1$. Then $|p_n| \le 2, n = \{1,2,3,\ldots\}$.

Lemma 2. [14]

Let the function $p \in P$ be given by the series (2). Then for any complex number σ,

$$|p_2 - \sigma p_1^2| \le 2\max\{1, |2\sigma - 1|\}$$

And the result is sharp for the functions given by
$p(z) = \frac{1+z}{1-z}$ $(z \in \mathbb{U})$.

Lemma 3. [15]

If $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$ is analytic function with positive real parts in $\mathbb{U}$, then

$$|p_2 - v p_1^2| \leq \begin{cases} -4v + 2, & if\, v \leq 0 \\ 2, & if\, 0 \leq v \leq 1 \\ 4v - 2, & if\, v \geq 1 \end{cases}$$

Opoola Differential Operator. [19]: For $\wp \geq 0, 0 \leq \mu \leq \beta, n \in \mathbb{N}_0, z \in \mathbb{U}$. the Opoola differential operator $D^n(\wp,\mu,\beta)f : A \to A$ is defined as follows

$$D^0(\wp,\mu,\beta)f(z) = f(z)$$

$$D^1(\wp,\mu,\beta)f(z) = \wp z f'(z) - z(\beta-\mu)\wp + (1 + (\beta - \mu\mu - 1)\wp)f(z)$$

$$D^n(\wp,\mu,\beta)f(z) = (D(D^{n-1}(\wp,\mu,\beta)f(z))) = z + \sum_{k=2}^{\infty}[1 + (k + \beta - \mu - 1)\lambda]^n a_k z^k \qquad (3)$$

This operator was studied by the author in [8].

Remark 1:

- When $\wp = 1$, $\mu = \beta$, then $D^n(\wp,\mu,\beta)f(z)$ becomes the Salagean differential operator.[22]
- When $\mu = \beta$, then $D^n(\wp,\mu,\beta)f(z)$ becomes the Al-Oboudi differential operator.[1]

Definition: For $t \in (0,1]$, a function $f(z)$ given by (2), analytic in the unit disk $\mathbb{U}$ is in the class $SC(n,b,\mu,\beta,\wp,U_n(t))$ if it satisfies the subordination condition below

$$1 + \frac{1}{b}\left\{\frac{z(D^n(\mu,\beta,\wp)f(z))'}{D^n(\mu,\beta,\wp)f(z)} - 1\right\} \prec H(z,t) \qquad (4)$$

Where $n \in \mathbb{N}_0 = \{0\} \cup \{1,2,3,\ldots\}, b \in \mathbb{C}/\{0\}, D^n(\mu,\beta,\wp)f(z)$ is given by equation (3) and $H(z,t)$ is givenby eq.(1).

Remark 2 :

when $H(z,t)$ is replaced with $\phi(z)$ (see [15]) in equation (4), then the following remarks hold

- When $n = 0$, equation (4) becomes the class $S_b^*(\phi)$ introduced by [21].
- When $n = 0,1$, $b = \wp = 1, \beta = \mu$, equation (4) become the classes of functions known as Ma-Minda star-like and convex functions respectively. See [15].
- When $n = 1$, $\beta = \mu, \wp = 1$, equation (4) becomes the class $C_b^*(\phi)$ introduced by [21].

III RESULTS AND DISCUSSIONS

COEFFICIENT BOUNDS FOR FUNCTIONS IN THE CLASS $SC(n,b,\mu,\beta,\wp,U_n(t))$

Theorem 1: Let the function $f \in A$ given by (2). If the function f is in the class $SC(n,b,\mu,\beta,\wp,U_n(t)), t \in (0,1], \theta \in [0,\pi]$, then

$$|a_2| \leq \frac{2t|b|}{[1 + (\beta - \mu + 1)\wp]^n}$$

$$|a_3| \leq \frac{1}{[1 + (\beta - \mu + 2)\wp]^n}\left|bt - \frac{1}{2}(b\{2t - 4t^2 + 1\} - b^2 4t^2\right|$$

$$|a_4| \leq \frac{1}{3[1 + (\beta - \mu + 3)\wp]^n}\left|t^3\{8b + 4b^3 + 12b^2\} - t(4b + 3b^2\right|$$

$$|a_5| \leq \frac{1}{4[1 + (\beta - \mu + 4)\wp]^n}\Big|t^4(72b^2 - 16b + 24b^4)$$
$$+ t^3(40b^2 + 48b - 32b^3) - t^2(76b^2 + 12b + 144b^3$$
$$- t(24b - 18b^2) - 36b^3 + \frac{37b^2}{2} + 5b\Big| \qquad (5)$$

Proof:

If $f(z) \in SC(n,b,\mu,\beta,\wp,U_n(t))$, then there exist a Schwarz function $\omega(z)$, analytic in the open unit disk $\mathbb{U}$ with $\omega(0) = 0, |\omega(z)| < 1$ such that from the definition of subordination and equation(4)

$$1 + \frac{1}{b}\left\{\frac{z(D^n(\mu,\beta,\wp)f(z))'}{D^n(\mu,\beta,\wp)f(z)} - 1\right\} = H(\omega(z),t)$$

From the definition of Chebyshev polynomial,

$$H(\omega(z),t) = 1 + U_1(t)(\omega(z)) + U_2(t)(\omega(z))^2 + U_3(t)(\omega(z))^3$$
$$+ U_4(t)(\omega(z))^4 + \cdots$$

Since $\omega(z)$ is a Schwarz function then the function $p(z)$ is defined as

$$p(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + p_1(z) + p_2 z^2 + \cdots$$

Which implies that

$$\omega(z) = \frac{p_1 z}{2} + \frac{1}{2}\left(p_2 - \frac{p_1^2}{2}\right)z^2 + \frac{1}{2}\left(p_3 - p_1 p_2 + \frac{p_1^3}{4}\right)z^3 + \frac{1}{2}\left(p_4 - p_1 p_3 - \frac{p_2^2}{2} + \frac{3p_1^2 p_2}{4} - \frac{p_1^4}{8}\right)z^4 + \cdots$$

Hence,

$$H(\omega(z),t)=\frac{U_1((t)p_1 z}{2}+\left[\frac{U_1((t)}{2}\left(p_2-\frac{p_1^2}{2}\right)+U_2((t)\frac{p_1^2}{4}\right]z^2$$
$$+\left[\frac{U_1((t)}{2}\left(p_3-p_1p_2+\frac{p_1^3}{4}\right)+\frac{U_2((t)}{2}\left(p_1p_2-\frac{p_1^3}{2}\right)+U_3((t)\frac{p_1^3}{8}\right]z^3$$
$$+\left[\frac{U_1((t)}{2}\left(p_4-p_1p_3-\frac{p_2^2}{2}+\frac{3p_1^2p_2}{4}-\frac{p_1^4}{8}\right)+\frac{U_2((t)}{2}\left(p_1p_3+\frac{p_2^2}{2}-\frac{3p_1^2p_2}{4}-\frac{3p_1^4}{8}\right)+\frac{U_3((t)}{8}\left(3p_1^2p_2-\frac{3p_1^4}{2}\right)\right.$$
$$\left.+\frac{U_4((t)p_1^4}{16}\right]z^4+\cdots \quad (5)$$

Also,

$$1+\frac{1}{b}\left\{\frac{z(D^n(\mu,\beta,\wp)f(z))'}{D^n(\mu,\beta,\wp)f(z)}-1\right\}$$
$$=\frac{1}{b}\{[1+(\beta-\mu+1)\wp]^n a_2 z+(2[1+(\beta-\mu+2)\wp]^n a_3-[1+(\beta-\mu+1)\wp]^{2n}a_2^2)z^2$$
$$+(3[1+(\beta-\mu+3)\wp]^n a_4-3[1+(\beta-\mu+1)\wp]^n[1+(\beta-\mu+2)\wp]^n a_2a_3$$
$$+[1+(\beta-\mu+2)\wp]^{3n}a_2^3)z^3$$
$$+(4[1+(\beta-\mu+4)\wp]^n a_5-4[1+(\beta-\mu+1)\wp]^n[1+(\beta-\mu+3)\wp]^n a_2a_4$$
$$-2[1+(\beta-\mu+1)\wp]^{2n}a_3^2+4[1+(\beta-\mu+1)\wp]^{2n}[1+(\beta-\mu+2)\wp]^n a_2^2a_3$$
$$-[1+(\beta-\mu+1)\wp]^{4n}a_2^4)z^4+\cdots\} \quad (6)$$

Comparing the coefficients of the powers of z in equations (5) and (6) and using lemma 1, the result of equation follows:

Corollary 1: if $f(z)\in SC(0,b,\mu,\beta,\wp,U_n(t)),t\in(0,1].then$ $|a_2|\le 2t|b|$

$$|a_3|\le 2t^2|(b+b^2|+\frac{1}{2}$$

$$|a_4|\le\frac{1}{3}|t^3\{8b+4b^3+12b^2\}-t(4b+3b^2|$$

$$|a_5|\le\frac{1}{4}\left|t^4(72b^2-16b+24b^4)+t^3(40b^2+48b-32b^3)-t^2(76b^2+12b+144b^3-t(24b-18b^2)-\right.$$
$$\left.+\frac{37b^2}{2}+5b\right|$$

Corollary 2:

When $SC(n,b,\mu,\beta,\wp,U_n(t))$ is reduced to $S(0,b,\phi(z))$,studied in [21].

We proceed to solve the Fekete-Szego problem for the new class of function defined when σ is complex

FEKETE-SZEGO INEQUALITY FOR FUNCTIONS IN THE CLASS $SC(n,b,\mu,\beta,\wp,U_n(t))$

Theorem 2: If $f(z)$ given by equation (2) belongs to the class of functions $SC(n,b,\mu,\beta,\wp,U_n(t))$, then for any complex number $\sigma,t\in(0,1],\theta\in[0,\pi]$

$$|a_3-\sigma a_2^2|\le\frac{bt}{[1+(\beta-\mu+2)\wp]^n}max\left[1,\left|\frac{4t^2-1}{2t}+\left(2\sigma\frac{[1+(\beta-\mu+2)\wp]^n}{[1+(\beta-\mu+1)\wp]^{2n}}-1\right)2bt\right|\right]$$

$$b\in\mathbb{C}\{0\},\quad 0\le\mu\le\beta,\quad \wp\ge 0,\quad \sigma\in\mathbb{C}$$

Proof:

From equation (5), the Fekete-Szego inequality can be written as

$$|a_3-\sigma a_2^2|\le\left|\frac{bt}{2[1+(\beta-\mu+2)\wp]^n}\{p_2-\nu p_1^2\}\right|$$

Where $\nu=\frac{1}{2}\left\{1-\frac{4t^2-1}{2t}+\left(2\sigma\frac{[1+(\beta-\mu+2)\wp]^n}{[1+(\beta-\mu+1)\wp]^{2n}}-1\right)2bt\right\}$ On application of lemmas 1 and 2, the result follows.

Corollary 3: .

If $f(z)\in SC(b,\phi(z))$, then

It reduces the class of functions studied in [21].

Next, we consider the case when δ is real in the following theorem

Theorem 3: Let $b>0,t\in(0,1]$ and let $f(z)$ be given by equation (2) belongs to the class of functions $SC(n,b,\mu,\beta,\wp,U_n(t))$. Then for $\delta\in\mathbb{R}$, we have

$$|a_3-\delta a_2^2|\le\begin{cases}k\left[-2\left\{1-\frac{4t^2-1}{2t}+(2\delta q-1)2bt\right\}+2\right] & if\ \delta\le\frac{1}{2q}\left\{\frac{4t^2-2t-1}{4bt^2}+1\right\}\\ 2k & if\ \frac{1}{2q}\left\{\frac{4t^2-2t-1}{4bt^2}+1\right\}\le\delta\frac{1}{2q}\left\{\frac{4t^2+2t-1}{4bt^2}+1\right\}\\ k\left[2\left\{1-\frac{4t^2-1}{2t}+(2\delta q-1)2bt\right\}-2\right] & if\ \delta\ge\frac{1}{2q}\left\{\frac{4t^2-2t-1}{4bt^2}+1\right\}\end{cases}$$

Where $k=\frac{1}{2[1+(\beta-\mu+2)\wp]^n}$

Proof:

Let $\delta\le\frac{1}{2q}\left\{\frac{4t^2-2t-1}{4bt^2}+1\right\}$ then from equation (5), the Fekete-Szego inequality can be written as

$$|a_3-\sigma a_2^2|\le\left|\frac{bt}{2[1+(\beta-\mu+2)\wp]^n}\{p_2-\nu p_1^2\}\right|$$

Where $\nu=\frac{1}{2}\left\{1-\frac{4t^2-1}{2t}+\left(2\delta\frac{[1+(\beta-\mu+2)\wp]^n}{[1+(\beta-\mu+1)\wp]^{2n}}-1\right)2bt\right\}$

On application of lemma 3, the following cases are considered;

Case 1:

$|a_2-\nu a_1^2|\le-4\nu+2, if\ \nu\le 0$ this implies that

$$|a_2-\nu a_1^2|=-4\nu+2=-4\left\{\frac{1}{2}\left\{1-\frac{4t^2-1}{2t}+\left(2\delta\frac{[1+(\beta-\mu+2)\wp]^n}{[1+(\beta-\mu+1)\wp]^{2n}}-1\right)2bt\right\}\right\}+2$$

That is when

$$\frac{1}{2}\left\{1-\frac{4t^2-1}{2t}+\left(2\delta\frac{[1+(\beta-\mu+2)\wp]^n}{[1+(\beta-\mu+1)\wp]^{2n}}-1\right)2bt\right\}\le 0,$$

Let $\frac{[1+(\beta-\mu+2)\wp]^n}{[1+(\beta-\mu+1)\wp]^{2n}}=q$, then

$$|a_3 - \delta a_2^2| \le k\left[-2\left\{1 - \frac{4t^2-1}{2t} + (2\delta q - 1)2bt\right\} + 2\right] \text{ if } \delta \le \frac{1}{2q}\left\{\frac{4t^2-2t-1}{4bt^2} + 1\right\} \quad (8)$$

Case 2:

$$|a_2 - \upsilon a_1^2| \le 2, \text{if } 0 \le \upsilon \le 1$$

If $0 \le \upsilon$, it implies that

$$\frac{1}{2}\left\{1 - \frac{4t^2-1}{2t} + \left(2\delta \frac{[1+(\beta-\mu+2)p]^n}{[1+(\beta-\mu+1)p]^{2n}} - 1\right)2bt\right\} \ge 0$$

Then $\delta \ge \frac{1}{2q}\left\{\frac{4t^2-2t-1}{4bt^2} + 1\right\}$

And if $\upsilon \le 1$, it implies that

$$\frac{1}{2}\left\{1 - \frac{4t^2-1}{2t} + \left(2\delta \frac{[1+(\beta-\mu+2)p]^n}{[1+(\beta-\mu+1)p]^{2n}} - 1\right)2bt\right\} \le 1$$

Which also implies that

$$\delta \le \frac{1}{2q}\left\{\frac{4t^2+2t-1}{4bt^2} + 1\right\}$$

So,

$$|a_3 - \delta a_2^2| \le 2k \text{ if } 0 \le \frac{1}{2q}\left\{\frac{4t^2-2t-1}{4bt^2} + 1\right\} \le \delta \le \frac{1}{2q}\left\{\frac{4t^2+2t-1}{4bt^2} + 1\right\} \quad (9)$$

Case 3:

$$|a_2 - \upsilon a_1^2| \le 4\upsilon - 2, \text{if } \upsilon \ge 1$$

then

$$|a_3 - \delta a_2^2| \le k\left[2\left\{1 - \frac{4t^2-1}{2t} + (2\delta q - 1)2bt\right\} - 2\right] \text{ if } \delta \ge \frac{1}{2q}\left\{\frac{4t^2-2t-1}{4bt^2} + 1\right\} \quad (10)$$

Hence, from equations (8), (9) and (10), the result of theorem 3 holds.

Conclusion: Opoola differential equation, see [19], when some conditions are set in, is a generalization of both Salagean and Al-Oboudi differential operators. This operator was used to define a new subclass of analytic and univalent functions. New results were obtained such as the proofs of theorem 1-3 and corollary 1. Also, the generalization of the existing results such as Remarks 1 and 2, corollaries 2and 3 were obtained in this work.

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