Weighted Inverted Generalized Exponential Distribution with an Application

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Abstract**:** This paper introduces the Weighted Inverted Generalized Exponential (WIGE) distribution byinducing inverted weight function into existing Inverted Generalized Exponential distribution. Various statistical properties of the proposed distribution were explicitly derived and the method of maximum likelihood estimation was used in estimating the model parameters. The model was applied to a real life data set and its performance was assessed with respect to Inverse Exponential (IE), Generalized Exponential (GE) and Inverted Generalized Exponential (IGE) distributions using the log-likelihood and Akaike Information Criteria as basis for judgment. The proposed distribution gives better fit when subjected to a positive skewed with extraneous variation lifetime dataset.

Keywords: Exponential distribution, Generalization, Inversion, Statistical Properties, Weighted distributions, Azzalini.

I. INTRODUCTION

The weibull and exponential distributions have been considered by many authors in modelling lifetime data. For instance, survival analysis in living and engineering sciences by Diamoutene, Abdoulaye et al. (2016), parametric regression model for survival data by Zhongheng Zhang (2016) and many more. However, more generalizations ofweibull and exponential distributions are still required for difference lifetime data behaviours.

Generalized exponential distribution introduced by Gupta and Kundu, (2000) has been studied extensively by several authors. Thisdistribution can be used as an alternative to gamma or weibull distribution.Oguntunde and Adejumo, (2015) proposed a two parameter Inverted Generalized Exponential (IGE) and a three parameter Generalized Inverted Generalized Exponential (GIGE) probability distributions as generalizations of the one-parameter Exponential distribution.They explored the statistical properties of the GIGE distribution and estimated its parameters using the method of maximum likelihood estimation (MLE).

Weighted Distribution

Suppose x is a non-negative random variable with probability density function $f\left(x\right)$

Then, the weighted density function $f_w\left(x\right)$ is defined as

$$f_w(x) = \frac{w(x) f(x)}{w_D}$$

where $w(x)$ is the weight function and $w_D = \int_0^\infty w(x) f(x) dx$

A random variable Xis said to have an Inverted Generalized Exponential distribution with parameters α *and* λ if its Probability Density function (PDF) and Cumulative Distribution Function (CDF) are given respectively by:

$$F_{IGE}(x) = 1 - \left(1 - e^{-\left(\frac{\lambda}{x}\right)}\right)^{\alpha}; x > 0, \alpha > 0, \lambda > 0 \quad 1$$

$$f_{IGE}(x) = \alpha\lambda e^{-\left(\frac{\lambda}{x}\right)} x^{-2} \left(1 - e^{-\left(\frac{\lambda}{x}\right)}\right)^{\alpha-1}; x > 0, \alpha > 0, \lambda > 0 \quad 2$$

II. MATERIALS AND METHODS

Let X denote a continuous random variable, considering the weight function $w(x) = x^{-1}$ and the two-parameter Inverted Generalized distribution as given in equation (1) and (2), then the pdf and cdf of the Weighted Inverted Generalized Exponential distribution are:

$$f_w(x) = \frac{\alpha\lambda^2}{\Psi(\alpha+1) - \Psi(1)} x^{-3} e^{-\frac{\lambda}{x}} \left(1 - e^{-\frac{\lambda}{x}}\right)^{\alpha-1} \quad x > 0, \alpha > 0, \lambda > 0 \quad 3$$

and

$$F_w(x) = \frac{\alpha e^{-\left(\frac{\lambda}{x}\right)} (x+\lambda) hypergeom\left(\left[1,1,1-\alpha, \frac{x+2\lambda}{\lambda}\right], \left[2,2, \frac{x+\lambda}{\lambda}\right], e^{-\left(\frac{\lambda}{x}\right)}\right)}{x\left(\Psi(\alpha+1) - \Psi(1)\right)} \quad x > 0, \alpha > 0, \lambda > 0 \quad 4$$

respectively.

where λ is a scale parameter and α is the shape parameter

Derivation of WIGE Distribution

$$f_w(x)=\frac{w(x)f(x)}{w_D}$$

where $f(x)$ is the pdf of IGE and $w_D=\int_0^\infty w(x)f(x)dx$

$$w_D=\int_0^\infty x^{-1}\alpha\lambda e^{-\left(\frac{\lambda}{x}\right)}x^{-2}\left(1-e^{-\left(\frac{\lambda}{x}\right)}\right)^{\alpha-1}dx \qquad 5$$

$$w_D=\int_0^\infty x^{-3}\alpha\lambda e^{-\left(\frac{\lambda}{x}\right)}\left(1-e^{-\left(\frac{\lambda}{x}\right)}\right)^{\alpha-1}dx$$

Let $y=\frac{\lambda}{x}$ then

$$w_D=\frac{\alpha}{\lambda}\int_0^\infty ye^{-y}\left(1-e^{-y}\right)^{\alpha-1}dy \qquad 6$$

where $\left(1-e^{-y}\right)^{\alpha-1}=\sum_{k=0}^{\alpha-1}(-1)^k\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-k)k!}e^{-yk}$

$$w_D=\frac{\alpha}{\lambda}\int_0^\infty ye^{-y}\sum_{k=0}^{\alpha-1}(-1)^k\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-k)k!}e^{-yk}dy \qquad 7$$

$$w_D=\frac{\alpha}{\lambda}\sum_{k=0}^{\alpha-1}(-1)^k\frac{\Gamma(\alpha)}{\Gamma(\alpha-k)k!}\int_0^\infty ye^{-y}e^{-yk}dy$$

$$w_D=\frac{\alpha}{\lambda}\sum_{k=0}^{\alpha-1}(-1)^k\frac{\Gamma(\alpha)}{\Gamma(\alpha-k)k!}\int_0^\infty ye^{-y(k+1)}dy$$

$$w_D=\frac{\alpha}{\lambda}\sum_{k=0}^{\alpha-1}(-1)^k\frac{\Gamma(\alpha)}{\Gamma(\alpha-k)k!(k+1)^2} \quad \text{where } \sum_{k=0}^{\alpha-1}(-1)^k\frac{\Gamma(\alpha)}{\Gamma(\alpha-k)k!(k+1)^2}=\frac{\Psi(\alpha+1)-\Psi(1)}{\alpha} \qquad 8$$

and $\Psi(\alpha+1)=\frac{d}{d\alpha}\Gamma(\alpha+1)$ which is known as the digamma function.

$$w_D=\frac{\alpha}{\lambda}\left[\frac{\Psi(\alpha+1)-\Psi(1)}{\alpha}\right] \qquad 9$$

$$w_D = \frac{1}{\lambda}\left[\Psi(\alpha+1) - \Psi(1)\right] \qquad 10$$

Therefore; $f_w(x) = \frac{w(x)f(x)}{w_D}$

$$f_w(x) = \frac{x^{-1}\alpha\lambda e^{-\left(\frac{\lambda}{x}\right)} x^{-2}\left(1-e^{-\left(\frac{\lambda}{x}\right)}\right)^{\alpha-1}}{\left[\frac{1}{\lambda}\left[\Psi(\alpha+1)-\Psi(1)\right]\right]}$$

$$f_w(x) = \frac{\alpha\lambda^2}{\Psi(\alpha+1)-\Psi(1)} x^{-3} e^{-\frac{\lambda}{x}}\left(1-e^{-\frac{\lambda}{x}}\right)^{\alpha-1} \qquad 11$$

Equation (11) is the pdf of the Weighted Inverted Generalized Exponential distribution.

Its associated cdf is obtained as follows:

$$F(x) = \int_0^x f(y)\,dy$$

$$F_w(x) = \frac{\alpha e^{-\left(\frac{\lambda}{x}\right)}(x+\lambda)\, hypergeom\left(\left[1,1,1-\alpha,\frac{x+2\lambda}{\lambda}\right],\left[2,2,\frac{x+\lambda}{\lambda}\right], e^{-\left(\frac{\lambda}{x}\right)}\right)}{x\left(\Psi(\alpha+1)-\Psi(1)\right)} \qquad 12$$

The hypergeom in the CDF is generalized hypergeometric function.

I. Proof of validity of WIGE Distribution

For the PDF to be valid, it suffices that; $\int_0^\infty f_w(x)\,dx = 1$

$$q\int_0^\infty e^{-\left(\frac{l}{x}\right)} x^{-3}\left(1-e^{-\left(\frac{l}{x}\right)}\right)^{a-1} dx = 1$$

$$\text{where } \theta = \frac{\alpha\lambda^2}{\Psi(\alpha+1)-\Psi(1)}$$

let $y = \frac{\lambda}{x}$ then

$$\frac{\theta}{\lambda^2}\int_0^\infty ye^{-y}\left(1-e^{-y}\right)^{\alpha-1}dy$$

where $\left(1-e^{-y}\right)^{\alpha-1} = \sum_{k=0}^{\alpha-1}(-1)^k\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-k)k!}e^{-yk}$

$$\frac{\theta}{\lambda^2}\int_0^\infty ye^{-y}\sum_{k=0}^{\alpha-1}(-1)^k\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-k)k!}e^{-yk}dy$$

$$\frac{\theta}{\lambda^2}\sum_{k=0}^{\alpha-1}(-1)^k\frac{\Gamma(\alpha)}{\Gamma(\alpha-k)k!}\int_0^\infty ye^{-y}e^{-yk}dy \qquad 13$$

$$\frac{\theta}{\lambda^2}\sum_{k=0}^{\alpha-1}(-1)^k\frac{\Gamma(\alpha)}{\Gamma(\alpha-k)k!}\int_0^\infty ye^{-y(\mathrm{k}+1)}dy$$

$$\frac{\theta}{\lambda^2}\sum_{k=0}^{\alpha-1}(-1)^k\frac{\Gamma(\alpha)}{\Gamma(\alpha-k)k!(\mathrm{k}+1)^2}$$

where $\sum_{k=0}^{\alpha-1}(-1)^k\frac{\Gamma(\alpha)}{\Gamma(\alpha-k)k!(\mathrm{k}+1)^2} = \frac{\Psi(\alpha+1)-\Psi(1)}{\alpha}$

$$\theta\left[\frac{\Psi(\alpha+1)-\Psi(1)}{\lambda^2\alpha}\right]$$

$$\left[\frac{\alpha\lambda^2}{\Psi(\alpha+1)-\Psi(1)}\right]\left[\frac{\Psi(\alpha+1)-\Psi(1)}{\lambda^2\alpha}\right]=1$$

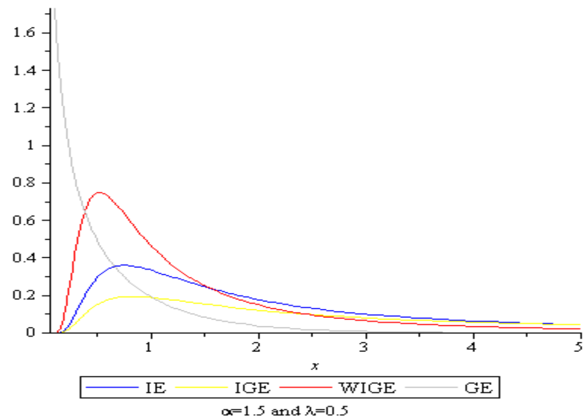

Figure 1: Plot of probability density function of IE, IGE, WIGE & GE distributions

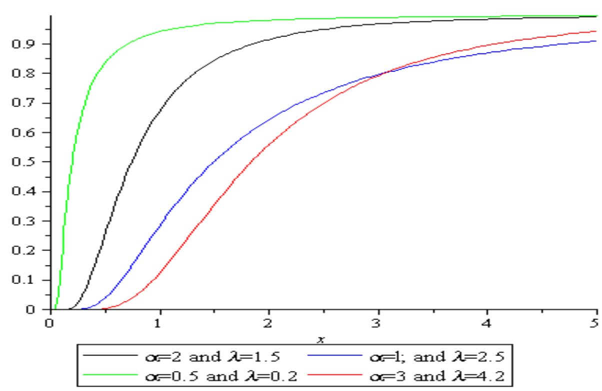

Figure 2: Plot of cumulative density function of WIGE

A. Reliability Analysis

Survival Function: The Survival function is given by:

$$S(x)=1-F(x)$$

$$S(x)=1-\left[\frac{\alpha e^{-\left(\frac{\lambda}{x}\right)}(x+\lambda)\,hypergeom\left(\left[1,1,1-\alpha,\frac{x+2\lambda}{\lambda}\right],\left[2,2,\frac{x+\lambda}{\lambda}\right],e^{-\left(\frac{\lambda}{x}\right)}\right)}{x\left(\Psi(\alpha+1)-\Psi(1)\right)}\right]$$

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Hazard function: The Hazard function is also given by:

$$h(x)=\frac{\theta x^{-3}e^{-\frac{\lambda}{x}}\left(1-e^{-\frac{\lambda}{x}}\right)^{\alpha-1}}{1-\left[\frac{\alpha e^{-\left(\frac{\lambda}{x}\right)}(x+\lambda)\,hypergeom\left(\left[1,1,1-\alpha,\frac{x+2\lambda}{\lambda}\right],\left[2,2,\frac{x+\lambda}{\lambda}\right],e^{-\left(\frac{\lambda}{x}\right)}\right)}{x\left(\Psi(\alpha+1)-\Psi(1)\right)}\right]} \quad \text{where} \quad \theta=\frac{\alpha\lambda^2}{\Psi(\alpha+1)-\Psi(1)}$$

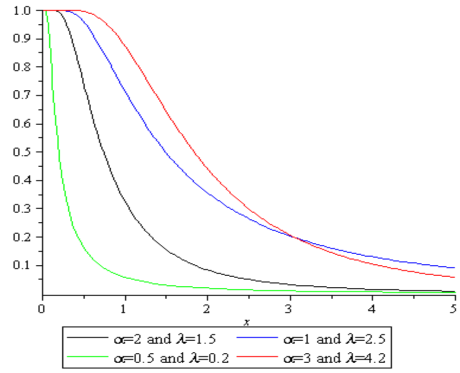

Figure 3: Survival plot of the WIGEl Distribution

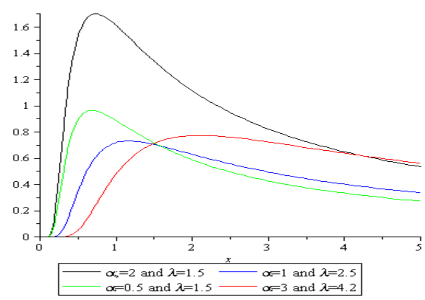

Figure 4: Hazard plot of the WIGE Distribution

B. Moment of WIGE Distribution

The moment of distribution is very important, it will help us to determine the mean, dispersion, coefficients of skewness and kurtosis. The kth moments of a non negative random variable X is defined as

$$E\left(X^r\right)=\int_0^\infty x^r f(x)\,dx$$

$$E\left(X^{r}\right)=\theta\int_{0}^{\infty}x^{r}x^{-3}e^{-\frac{\lambda}{x}}\left(1-e^{-\frac{\lambda}{x}}\right)^{\alpha-1}dx \qquad 15$$

where $\theta=\dfrac{\alpha\lambda^{2}}{\Psi(\alpha+1)-\Psi(1)}$

Taking $y=\dfrac{\lambda}{x}$

$$E\left(X^{r}\right)=\theta\int_{0}^{\infty}\left(\frac{\lambda}{y}\right)^{r}\left(\frac{y}{\lambda}\right)^{3}\left(\frac{\lambda}{y}\right)^{2}\left(\frac{1}{\lambda}\right)e^{-y}\left(1-e^{-y}\right)^{\alpha-1}dy$$

$$E\left(X^{r}\right)=\frac{\theta}{\lambda^{2}}\int_{0}^{\infty}\left(\frac{\lambda}{y}\right)^{r}ye^{-y}\left(1-e^{-y}\right)^{\alpha-1}dy$$

$$E\left(X^{r}\right)=\frac{\theta\lambda^{r}}{\lambda^{2}}\int_{0}^{\infty}y^{1-r}e^{-y}\left(1-e^{-y}\right)^{\alpha-1}dy$$

where $\left(1-e^{-y}\right)^{\alpha-1}=\sum_{k=0}^{\alpha-1}(-1)^{k}\dfrac{\Gamma(\alpha-1)}{\Gamma(\alpha-k)k!}e^{-yk}$

$$E\left(X^{r}\right)=\frac{\theta\lambda^{r}}{\lambda^{2}}\sum_{k=0}^{\alpha-1}(-1)^{k}\frac{\Gamma(\alpha)}{\Gamma(\alpha-k)k!}\int_{0}^{\infty}y^{1-r}e^{-y}e^{-yk}dy$$

$$E\left(X^{r}\right)=\frac{\theta\lambda^{r}}{\lambda^{2}}\sum_{k=0}^{\alpha-1}(-1)^{k}\frac{\Gamma(\alpha)}{\Gamma(\alpha-k)k!}\int_{0}^{\infty}y^{1-r}e^{-y(k+1)}dy$$

Let $v=y(k+1)$

$$E\left(X^{r}\right)=\frac{\theta\lambda^{r}}{\lambda^{2}}\sum_{k=0}^{\alpha-1}(-1)^{k}\frac{\Gamma(\alpha)}{\Gamma(\alpha-k)k!(k+1)^{2-r}}\int_{0}^{\infty}v^{1-r}e^{-v}dv$$

$$E\left(X^{r}\right)=\frac{\theta\lambda^{r}}{\lambda^{2}}\sum_{k=0}^{\alpha-1}(-1)^{k}\frac{\Gamma(\alpha)\Gamma(2-r)}{\Gamma(\alpha-k)k!(k+1)^{2-r}}$$

Substituting $\theta=\dfrac{\alpha\lambda^{2}}{\Psi(\alpha+1)-\Psi(1)}$

$$E\left(X^{r}\right)=\frac{\alpha\lambda^{r}}{\Psi(\alpha+1)-\Psi(1)}\sum_{k=0}^{\alpha-1}(-1)^{k}\frac{\Gamma(\alpha)\Gamma(2-r)}{\Gamma(\alpha-k)k!(k+1)^{2-r}} \qquad 16$$

C. Moment Generating Function of WIGE Distribution

Following (Cordeiro, 2011) the expression for moment generating function is given as;

$$M_x(t)=\sum_{r=0}^{n}\left[\frac{t^r}{r!}\frac{\alpha\lambda^r}{\Psi(\alpha+1)-\Psi(1)}\sum_{k=0}^{\alpha-1}(-1)^k\frac{\Gamma(\alpha)\Gamma(2-r)}{\Gamma(\alpha-k)k!(k+1)^{2-r}}\right]$$

The moment generating function is the expected value of exponential function of tX, i.e, the moment generating function of random variable X is given as:

$$M_x(t)=E\left(e^{tX}\right)$$

$$\text{Where } E\left(e^{tX}\right)=\int_0^\infty e^{tX} f(x)\,dx$$

with the use of Taylor's series

$$M_x(t)=\int_0^\infty\left(1+\frac{tx}{1!}+\frac{t^2x^2}{2!}+\ldots+\frac{t^rx^r}{r!}\ldots\right)f(x)\,dx$$

$$M_x(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}E\left(X^r\right)$$

and $E\left(X^r\right)$ is defined in Equation (16) above, then

$$M_x(t)=\sum_{r=0}^{n}\left[\frac{t^r}{r!}\frac{\alpha\lambda^r}{\Psi(\alpha+1)-\Psi(1)}\sum_{k=0}^{\alpha-1}(-1)^k\frac{\Gamma(\alpha)\Gamma(2-r)}{\Gamma(\alpha-k)k!(k+1)^{2-r}}\right] \qquad 17$$

D. Parameter Estimation of WEIE distribution

The Estimation of Weighted Inverted Generalized Exponential distribution is obtained using the Method of Maximum Likelihood Estimation (MLE). The formula of MLE contains the unknown parameters of the distribution. The values of these parameters that maximize the sample likelihood are known as the ML estimates (Elgarhy, 2017)

Let $x_1, x_2, \ldots, x_n$ be a random sample of size "n" from Weighted Inverted Generalized distribution defined in equation (3) and (4), then Likelihood function $L(x/\alpha,\beta)$ is given by

$$L(x/\alpha,\lambda)=\prod_{i=1}^{n} f(\mathrm{x}_i/\alpha,\lambda)$$

Let $l = \log L(x/\alpha,\lambda)$

$$l(\alpha,\lambda) = 2n\log\lambda + n\log\alpha - 3\sum_{i=1}^{n}\log(x_i) - \lambda\sum_{i=1}^{n}\frac{1}{x_i} + (\alpha-1)\sum_{i=1}^{n}\log\left(1-e^{-\frac{\lambda}{x_i}}\right) - n\log\left(\Psi(\alpha+1)-\Psi(1)\right) \quad 18$$

Differentiating equation (18) with respect to α

$$\frac{dl(\beta)}{d\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left(1-e^{-\frac{\lambda}{x_i}}\right) - n\left[\frac{\Psi(1,\alpha+1)}{\Psi(\alpha+1)-\Psi(1)}\right] \quad 19$$

Where $\Psi(1,\alpha+1) = \frac{d^2}{d\alpha^2}\Gamma(\alpha+1)$ is known as tri-gamma function

Differentiating equation (18) with respect to λ

$$\frac{dl(\beta)}{d\lambda} = \frac{2n}{\lambda} - \sum_{i=1}^{n}\frac{1}{x_i} + (\alpha-1)\sum_{i=1}^{n}\left[\frac{e^{-\left(\frac{\lambda}{x_i}\right)}}{x_i\left(1-e^{-\left(\frac{\lambda}{x_i}\right)}\right)}\right] \quad 20$$

Setting equation (19) and (20) to zero and solving the resulting non-linear equations simultaneously will give the maximum likelihood estimates of parameters α and λ

II. Application to Data Set

The application to real life data set of the Weighted Inverted Generalized Exponential distribution is provided. The performance of the WIGE distribution was compared with that of existing Inverse Exponential (IE), Generalized Exponential (GE) and Inverted Generalized Exponential (IGE) distributions using log-likelihood and Akaike Information Criterion as selection criteria. The distribution that corresponds to the highest log-likelihood value and lowest AIC value is selected as the best for the data set used.

Data Set: Fertility is a key component that determines size of a household and nation population. Fertility analysis important for policymakers to conclude guidance for population control and also for the evaluation of performance of family planning programmes. The birth interval of the second child is defined as difference in months between first birth and second birth.

The Data set used for further analysis in this research is secondary data obtained from Demography and Health Survey 2015. Data set contains the length (months) of the preceding birth interval of second birth from 532 women from South East of Nigeria where the household heads are female

Table 1: Summary of length (months) of the preceding birth interval for second birth.

N	Mean	Med.	Var.	Skewness	Kurtosis
532	40.34	32.00	876.0415	4.438698	37.34944

Table 2: Analysis of the performance of the competing distributions

Models	Estimates	LL	AIC
WIGE	$\hat{\alpha} = 4.0624(0.2632)$ $\hat{\lambda} = 83.8563(0.7673)$	-2284.456	4572.912
IGED	$\hat{\alpha} = 5.1325(0.2018)$ $\hat{\lambda} = 69.5320(NA)$	-2285.335	4574.669
GE	$\hat{\alpha} = 5.0946(0.4567)$ $\hat{\lambda} = 0.0580(0.0027)$	-2321.9880	4647.975
IE	$\hat{\lambda} = 30.3300(1.0490)$	-2480.962	4963.923

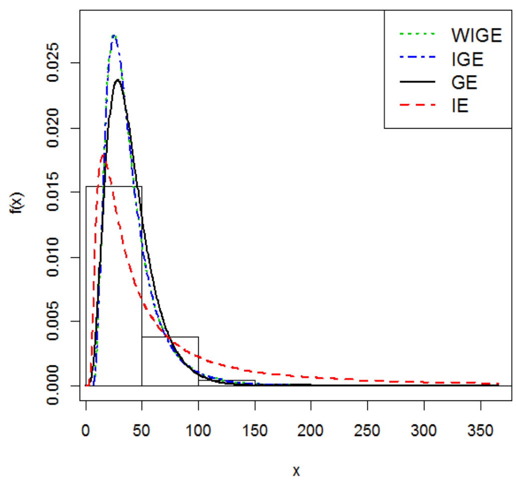

Figure 5 Histogram with competing distributions on birth interval data set

Remarks: Table 1 reveals that the dataset considered for this study is positively skewed with extraneous variation(mean = 40.34 and variance = 876.0415). Also, From Table 2, the WIGE distribution has the highest log-likelihood value and the lowest AIC value, therefore, it can be concluded that it more suitable for the data set than other existing distributions.

III. CONCLUSION

The Weighted version of the Inverted Generalized Exponential distribution has been successfully derived. The model has unimodal (inverted bathtub) and decreasing shapes (depending on the value of the parameters). Explicit expressions for its basic statistical properties have been successfully derived. The model exhibits unimodal and decreasing failure rates, this implies that the model can be used to describe and model real life phenomena with unimodal or decreasing failure rates. For the real life application provided, the Weighted Inverted Generalized Exponential distribution performs better than other existing distributions; it is however a good and competitive model.

Conflict of Interest

The authors declare no conflict of interest

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